(2)

AD-A223 615

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	4/24/90	Progress Report (8/1/88-7/31/89)

4. TITLE AND SUBTITLE
Theoretical and Experimental Research into Biological Mechanisms Underlying Learning and Memory

5. FUNDING NUMBERS
2305/B4

6. AUTHOR(S)
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8. PERFORMING ORGANIZATION REPORT NUMBER
AFOSR-TR- 90 0672

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Dept. of the Air Force
Air Force Office of Scientific Research (AFSC)
Bolling Air Force Base, DC 20332-6448
Abn Craig

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
AFOSR 88-0228

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

We describe an extended model of backward propagation incorporating gain modification and compare the performance of the extended model with ordinary backward propagation.

We also describe our work on a statistical model for feature extraction based on the BCM neural network model (Bienenstock, Cooper, and Munro, 1982). The model is presented as an exploratory (Projection Pursuit) algorithm (PP). The formulation, which is similar in nature to PP, is based on a minimization of a cost function over a set of parameters, yielding an optimal decision rule under some norm. We have presented a new projection index (cost function) that favors directions possessing multi-modality, where the multi-modality is measured in terms of the separability property of the data. The synaptic modification equations, which perform the minimization of the cost function, turn out to be similar to the synaptic modification equations governing learning in BCM neurons.

14. SUBJECT TERMS	15. NUMBER OF PAGES	16. PRICE CODE
	6	

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	Unlimited

NSN 7540-01-280-5500 Standard Form 298 (Rev 2-89)
Prescribed by ANSI Std 239-18
298-102

Theoretical and Experimental Research into Biological Mechanisms Underlying Learning and Memory

AFOSR Grant # 88-0228

Final Report

April 1990

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List of Publications Supported

An Averaging Result for Random Differential Equations, Nathan Intrator, Technical Report, Center for Neural Science, Brown University, April 9, 1990.

Gain Modification in a Backward Propagation Neural Network, Charles M. Bachmann, Ph. D. dissertation, May, 1990, (in preparation), to be published by UMI Dissertation Services.

GAIN MODIFICATION IN A BACKWARD PROPAGATION NEURAL NETWORK

Our research into backward propagation has led to a number of new theoretical and empirical results. We have developed a generalized version of backward propagation which incorporates gain modification. In our generalized network, both gains and synapses are modified by a backward propagation procedure. Synapses are modified in proportion to the negative gradient of the energy with respect to the synaptic weight as in ordinary backward propagation, and gains are modified in proportion to the negative partial derivative with respect to the gain. Since the resulting error signals for the gain and synaptic weights are proportional to one another, the computational complexity of our generalized network is comparable to that of the original backward propagation model.

Simulations of the new network have been performed on a concentric circle paradigm in two-dimensions. In the concentric circle problem, we present the x and y coordinates of patterns in the unit circle. Those patterns which lie outside of a pre-determined radius are in one class, while those interior to the radius belong to a second class. In our technical report "Gain Modification Enhances High Momentum Backward Propagation" (Bachmann, 1989) we demonstrated that a combination of high momentum and gain modification leads to faster convergence rate compared with high momentum alone. Bare backward propagation converged at an even slower rate, as expected. The definition of convergence for this study was that the network response for all patterns fall within 0.1 of the target output . Additional work which we have carried out since the publication of our report has shown that the onset of generalization for this paradigm actually occurs on fairly short time scales, and there is essentially little difference in generalization between momentum and momentum with gain modification on short time scales. However, both of these approaches achieve significantly better levels of generalization than bare backward propagation on short time scales. In essence, we have shown that with momentum or a combination of gain modification, the network learns to generalize rapidly compared to ordinary backward propagation. However, in precisely fitting the training data, the best convergence rate is achieved by a combination of gain modification and momentum.

STATISTICAL FORMULATION OF FEATURE EXTRACTION

Our mathematical analysis of unsupervised learning has led to the statistical formulation of the parameter estimation problem associated with unsupervised learning in a neural network. The network is presented as an exploratory projection pursuit method that

performs feature extraction (or dimensionality reduction) on the training data set. The formulation, which is similar in nature to PP, is based on a minimization of a cost function over a set of parameters, yielding an optimal decision rule under some norm.

We have presented a new projection index (cost function) that favors directions possessing multi-modality, where the multi-modality is measured in terms of the separability property of the data. The synaptic modification equations, which perform the minimization of the cost function, turn out to be similar to the synaptic modification equations governing learning in BCM neurons (Bienenstock, Cooper, and Munro 1982). This has led to a new statistical viewpoint on the biologically-inspired BCM neuron, making it a plausible candidate for statistical feature extraction. The directions (synaptic weights) sought by the neuron maximize some kind of skewness measure of the projected distribution in this direction, which is one of the measures of deviation from normality, and therefore a direction which discovers an important structure of the high-dimensional data.

A network was presented based on the multiple feature extraction formulation. Both the linear and non-linear neurons were analyzed.

Part of the analysis of the synaptic modification equations, which are stochastic in nature due to the random inputs, was to compare their trajectories with a deterministic differential equation. The deterministic equation corresponds to the average (expected value) of the random differential equation, and is much easier to handle since it was shown that it represents the gradient of our projection index.

The connection between the synaptic modification equations and their deterministic version was analyzed by extending a general result on random differential equations (Geman, SIAM 1979). This work concerns differential equations which contain strong mixing random process. Mixing roughly says how the future of a random process depends on its past. The solution process is shown to be well approximated in a probabilistic sense by a deterministic trajectory, over infinite time interval, using the interplay between the rate of fluctuations of the random process, and the rate of the mixing.

Gain Modification in a Backward Propagation Neural Network

by

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A.B., Princeton University, 1984

Sc.M., Brown University, 1986

Thesis

Submitted in partial fulfillment of the requirements for the
Degree of Doctor of Philosophy in the Department of
Physics at Brown University

May 1990

AN AVERAGING RESULT FOR RANDOM DIFFERENTIAL EQUATIONS

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Abstract This paper concerns differential equations which contain strong mixing random processes. The solution process is shown to be well approximated by a deterministic trajectory, over an infinite time interval, using the interplay between the rate of fluctuations of the random process and the rate of the φ mixing. An application of the result is given for analysing synaptic modifications in Neural Networks.

1. Introduction

The mathematical theory of stochastic differential equations is concerned mainly with the study of Itô equations and the associated Markov process. Mostly, the results on non Itô type equations have been concerned with the conditions under which $x_\epsilon(t)$ converges (as $\epsilon \to 0$) to a diffusion process on finite intervals $[0, T/\epsilon]$ (cf. Stratonovich, 1963; Cogburn and Hersh, 1973; Papanicolaou and Kohler, 1974; Blankenship and Papanicolaou 1977). Averaging results for random differential equations are usually discussed in conjunction with the law of large numbers Kohler and Papanicolaou (1976) with the central limit theorem for $(x_\epsilon(t) - y_\epsilon(t))/\sqrt{\epsilon}$ on $[0, T]$ (cf. Khasminskii, 1966; and White 1976). Geman (1979) showed that the solution process of a random differential equation which contains strong mixing random process is well approximated by a deterministic trajectory over a finite time interval, and for a more restricted systems, over the infinite time interval. Analysis

analogous to that was carried out on Itô type equations by Vrkoc (1966), and by Lybrand (1975).

In this paper we shall continue the direction taken by Geman and approximate the solution process by a deterministic trajectory over an infinite time interval, using the interplay between the rate of fluctuations of the random process and the rate of the φ mixing, yielding a result for a wide family of nonlinear random differential equations. We will establish conditions under which the random solution *stays close* in L^2 sense to the associated deterministic solution. The result is particularly useful when a converging deterministic equation is approximated by a random equation that is more computationally feasible. Section 4 is devoted to such an application, in the theory of synaptic modification in Neural Networks.

Similar analysis was carried out on the discrete time version of such equations, see Ljung (1978), Kushner and Clark, (1978), Dupuis and Kushner (1987), and the references therein.

2. Formulation and statement of the problem

In this section we briefly summarize the relevant results form Geman (1977, 1979).

Let $\phi(t,\omega)$ be a bounded stationary stochastic process with $\mathcal{F}_0^t$ and $\mathcal{F}_t^\infty$ the σ-fields generated by $\{\phi(\tau,\omega) : 0 \le \tau \le t\}$, and $\{\phi(\tau,\omega) : t \le \tau < \infty\}$ respectively. Let the signed measure $v_{t,\delta}$ be defined on $(\Omega \times \Omega, \mathcal{F}_0^t \times \mathcal{F}_{t+\delta}^\infty)$ by

$$v_{t,\delta} = P(\omega : (\omega,\omega) \in B) - P \times P(B), \quad \text{for } B \in \mathcal{F}_0^t \times \mathcal{F}_{t+\delta}^\infty.$$

For any $\{B \in \mathcal{F}_0^t \times \mathcal{F}_{t+\delta}^\infty\}$, the set $\left(\omega : (\omega,\omega) \in B\right)$ is in $\mathcal{F}$, and since it is also a monotone class, v is well defined. The stochastic process $\phi(t,\omega)$, is said to have Type II φ mixing if

$$\varphi(\delta) = \sup_{t \ge 0} \sup_{A \in \mathcal{F}_0^t \times \mathcal{F}_{t+\delta}^\infty} | v_{t,\delta}(A) | \underset{\delta \to \infty}{\longrightarrow} 0.$$

Remark on φ mixing: The results we describe hold for Type I mixing as well, both of which were introduced by Volkonskii and Rozanov (1959), since for both types of mixing we have $|v|_{t,\delta}(\Omega \times \Omega) \le 2\varphi(\delta)$.

Let ϵ be a positive number, and consider the system:

$$\begin{aligned} \dot{x}_\epsilon(t,\omega) &= H(x_\epsilon(t,\omega),\omega,t/\epsilon), \\ \dot{y}_\epsilon(t) &= G_\epsilon(y_\epsilon(t),t), \\ x_\epsilon(0,\omega) &= y_\epsilon(0) = x_0 \in R^n. \end{aligned} \tag{2.1}$$

Assume:

1. H is jointly measurable with respect to its three arguments.
2. $G_\epsilon(x,t) = E[H(x(s,\omega),t/\epsilon)]$, and for all i and j

$$\frac{\partial}{\partial x_j} G_i(x,t) \text{ exists, and is continuous in } (x,t).$$

3. For some $T > 0$:

a. There exists a unique solution, $x(t,\omega)$, on $[0,T]$ for almost all ω; and

b. A solution to

$$\frac{\partial}{\partial t} g(t,s,x) = G(g(t,s,x),t), \quad g(s,s,x) = x,$$

exists on $[0,T] \times [0,T] \times R^n$.

The following notations will be used:

1. $H_\epsilon(x_\epsilon(t,\omega),\omega,t) \stackrel{\text{def}}{=} H(x_\epsilon(t,\omega),\omega,t/\epsilon)$
2. $g_s(t,s,x) = (\partial/\partial s) g(t,s,x)$.
3. $g_x(t,s,x) =$ the $n \times n$ matrix with (i,j) component $(\partial/\partial x_j) g_i(t,s,x)$.
4. For $H(x,\omega,\tau)$ define the families of σ-fields $\mathcal{F}_0^t$ and $\mathcal{F}_t^\infty$ such that, for each $t \geq 0$, $\mathcal{F}_0^t$ contains the σ-field generated by

$$\{H(x,\omega,\tau) : 0 \leq \tau \leq t, x \in R^n\},$$

and $\mathcal{F}_t^\infty$ contains the σ-field generated by

$$\{H(x,\omega,\tau) : t \leq \tau < \infty, x \in R^n\}.$$

The relation between the random differential equation and its averaged version for system (2.1) under conditions (1), (2), and (3) is given by:

Lemma (Geman 1977) For any C^1 function $K : R^n \to R^1$ and $t \in [0, T)$:

$$E[K(x(t))] = K(y(t)) + \int_0^t \int_{\Omega\times\Omega} \Big(\frac{\partial}{dx} K(g(t,s,x(s,\omega)))\Big) \cdot H(x(s,\omega),\eta,s) d\upsilon_{s,0} ds,$$

provided that

$$\Big(\frac{\partial}{\partial x} K(g(t,s,x(s,\omega)))\Big) \cdot H(x(s,\omega),\eta,s), \quad \text{and}$$
$$\Big(\frac{\partial}{\partial x} K(g(t,s,x(s,\omega)))\Big) \cdot H(x(s,\omega),\omega,s)$$

are absolutely integrable on $\Omega \times \Omega \times [0,T]$, with respect to $dP(\omega)dP(\eta)ds$.

The proof of the lemma is based on the relationship between the initial conditions in time and in space for an ODE, namely: If $g(t,s,x)$ is the function satisfying

$$\frac{\partial}{\partial t} g(t,s,x) = G(g(t,s,x),t)$$

then

$$g_s(t,s,x) = -g_x(t,s,x)G(x,s)$$

for all $t \in [0,\infty)$, $s \in [0,\infty)$, and $x \in R^n$. This follows from the observation that $g(t,s,x)$ is constant along trajectories of the form $(s, x(s))$ (cf. Hartman, 1964 chap 5).

Theorem (Geman, 1977) Finite time averaging. Assume also that:

4. There exist continuous functions $B_1(r,t), B_2(r,t)$, and $B_3(r,t)$, such that for all $i,j,k,\tau \geq 0$, and ω:

   a. $\mid H_i(x,\omega,t,\tau) \mid \leq B_1(\mid x \mid, t)$;

   b. $\mid (\partial/\partial x_j) H_i(x,\omega,t,\tau) \mid \leq B_2(\mid x \mid, t)$;

   c. $\mid (\partial^2/\partial x_j \partial x_k) H_i(x,\omega,t,\tau) \mid \leq B_3(\mid x \mid, t)$.

5. $\sup_{\epsilon>0, t\in[0,T]} \mid y_\epsilon(t) \mid \leq B_4$ for some B_4 and T.

Then

$$\sup_{t\in[0,T]} \mid x_\epsilon(t) - y_\epsilon(t) \mid \underset{\epsilon\to 0}{\longrightarrow} 0$$

in probability.

3. Averaging on $[0,\infty)$

When averaging on an infinite interval we require that ϵ be a function of t and $\epsilon \searrow 0$, meaning that the mixing rate becomes stronger in time. More specifically, let ϵ be a function of the form $\epsilon(t) = \epsilon_0\bar{\epsilon}(t)$ where $\bar{\epsilon}$ is monotonically decreasing to zero in time.

The above lemma still holds when x, H, g and G are replaced by x_ϵ, H_ϵ, g_ϵ and G_ϵ respectively, and also when ϵ becomes a function of t.

In order for the approximation to hold on $[0,\infty)$ we require that B_1, B_2, B_3 are constants in condition 4 (this will be relaxed later) extend condition 5 to hold for $t \in [0,\infty)$, and add the following relation between the rate of the mixing of H and the convergence of ϵ to zero:

6. $\exists\ \gamma > 0,\ c > 0$, such that $\varphi(\delta) \le \delta^{-\gamma}$, and $\bar{\epsilon}(t) \le t^{-(\frac{1}{\gamma}+1+c)}$, for a monotone decreasing $\bar{\epsilon}$.

Theorem 3.1 Assume H_ϵ is of Type II φ mixing, and satisfies condition 1-6. then

$$\lim_{\epsilon_0\to 0} \sup_{t\ge 0} E \mid x_\epsilon(t) - y_\epsilon(t) \mid^2 = 0.$$

Proof: Assume first that t is an integer. Fix ϵ_0 and apply the lemma to the system using $K(x) = \mid x - y_\epsilon(t) \mid^2$:

$$E \mid x_\epsilon(t) - y_\epsilon(t) \mid^2 =$$
$$= \mid \int_0^t \int_{\Omega\times\Omega} \Big(\frac{\partial}{\partial x} K(g_\epsilon(t,s,x_\epsilon(s,\omega)))\Big) \cdot H_\epsilon(x_\epsilon(s,\omega),\eta,s) d\nu_{s,0} ds \mid$$
$$\le \sum_{k=1}^{\infty} \mid \int_{k-1}^{k} \int_{\Omega\times\Omega} \Big(\frac{\partial}{\partial x} K(g_\epsilon(t,s,x_\epsilon(s,\omega)))\Big) \cdot H_\epsilon(x_\epsilon(s,\omega),\eta,s) d\nu_{s,0} ds \mid$$

For any fixed $\delta_k > 0$ (to be chosen later), since each integral is bounded we can write $\forall k$:

$$\int_{k-1}^{k}\int_{\Omega\times\Omega}\Big(\frac{\partial}{\partial x}K(g_\epsilon(t,s,x_\epsilon(s,\omega)))\Big)\cdot H_\epsilon(x_\epsilon(s,\omega),\eta,s)dv_{s,0}ds =$$

$$I \quad = \int_{k-1}^{k-1+\delta_k}\int_{\Omega\times\Omega}\Big(\frac{\partial}{\partial x}K(g_\epsilon(t,s,x_\epsilon(s,\omega)))\Big)\cdot H_\epsilon(x_\epsilon(s,\omega),\eta,s)dv_{s,0}ds$$

$$II \quad + \int_{k-1+\delta_k}^{k}\int_{\Omega\times\Omega}\Big(\frac{\partial}{\partial x}K(g_\epsilon(t,s,x_\epsilon(s-\delta_k,\omega)))\Big)\cdot$$

$$H_\epsilon(x_\epsilon(s-\delta_k,\omega),\eta,s)dv_{s,0}ds$$

$$+ \int_{k-1+\delta_k}^{k}\int_{\Omega\times\Omega}\{\Big(\frac{\partial}{\partial x}K(g_\epsilon(t,s,x_\epsilon(s,\omega)))\Big)\cdot H_\epsilon(x_\epsilon(s,\omega),\eta,s)$$

$$III \quad - \Big(\frac{\partial}{\partial x}K(g_\epsilon(t,s,x_\epsilon(s-\delta_k,\omega)))\Big)\cdot H_\epsilon(x_\epsilon(s-\delta_k,\omega),\eta,s)\}dv_{s,0}ds.$$

The bounds on x_ϵ and its derivatives, and the smoothness of K imply that I is $O(\delta_k)$. In the second term we can replace $v_{s,0}$ by $v_{s-\delta,\delta}$ since these measures agree on $(\Omega\times\Omega, \mathcal{F}_0^{s-\delta}\times\mathcal{F}_s^\infty)$, $s \geq \delta$, and since $x_\epsilon(s-\delta,\omega)$ is $\mathcal{F}_0^{s-\delta}$ measurable. Since $v_{t,\delta}$ is the difference of two probability measures, the total variation measure satisfies:

$$|v|_{t,\delta}(\Omega\times\Omega) \leq 2, \text{ and } |v|_{t,\delta}(\Omega\times\Omega) = 2\sum_{A\in\mathcal{F}_0^t\times\mathcal{F}_{t+\delta}^\infty}|v|_{t,\delta}(A),$$

therefore, with Type II (or I) mixing: $|v|_{t,\delta}(\Omega\times\Omega) \leq 2\varphi(\delta)$. Applying this to the second integral and using the above bounds again we get that II is $O\Big(\varphi(\delta_k/\epsilon(k-1))\Big)$. The last term is also $O(\delta_k)$ from the smoothness of H_ϵ and of x_ϵ.

Now choose $\delta_k = \sqrt{\epsilon_0}(k-1)^{-(1+\frac{1}{2}c)}$, $k > 1$, then since $\epsilon(k-1) \leq \epsilon_0(k-1)^{-(\frac{1}{\gamma}+1+c)}$, we get $\delta_k/\epsilon(k-1) \geq \frac{1}{\sqrt{\epsilon_0}}(k-1)^{\frac{1}{\gamma}+\frac{1}{2}c}$. From the condition on φ we have $\varphi(\delta_k/\epsilon(k-1)) \leq \epsilon_0^{\frac{1}{2}\gamma}(k-1)^{-(1+\frac{1}{2}\gamma c)}$. Since $\gamma > 0$, the sum

$$\sum_{k>1}O(\delta_k) + O\Big(\varphi(\delta_k/\epsilon(k-1))\Big) = O\Big(\epsilon_0^{\frac{1}{2}(1+\gamma)}\Big).$$

For the segment of t between two integers, an analogous argument is applied yielding an extra term of the form $O(\epsilon_0^{\frac{\gamma}{2}}+\epsilon_0^{\frac{1}{2}})$. therefore $E\mid x_\epsilon(t)-y_\epsilon(t)\mid^2 = O\Big(\epsilon_0^{\frac{1}{2}(1+\gamma)}\Big)$ uniformly in t.

This implies that

$$\sup_{t\geq 0} E \mid x_\epsilon(t) - y_\epsilon(t) \mid^2 = O\left(\epsilon_0^{\frac{1}{2}(1+\gamma)}\right),$$

$$\lim_{\epsilon_0 \to 0} \sup_{t\geq 0} E \mid x_\epsilon(t) - y_\epsilon(t) \mid^2 = 0.$$

$\Diamond$

The following problem is closely related: For fixed ω, let $H(x,\omega,t)$ map $R^n \times R^m \times R^1$ into R^n. Assume that for each x, $H(x,\omega,t)$ is a mixing process, and for each x and t define $G(x,t) = E[H(x,\omega,t)]$. Consider the random equation

$$\dot{x}_{\bar{\epsilon}}(t,\omega) = \bar{\epsilon} H(x_{\bar{\epsilon}}(t,\omega),\omega,t), \quad x_{\bar{\epsilon}}(0,\omega) = x_0, \tag{3.1}$$

with its averaged equation

$$\dot{y}_{\bar{\epsilon}}(t) = \bar{\epsilon} G(y_{\bar{\epsilon}}(t),t), \quad y_{\bar{\epsilon}}(0) = x_0. \tag{3.2}$$

For equation (3.2) condition 6 becomes:

6′. $\exists \gamma > 0$, such that

i) $\varphi(\delta) < \delta^{-\gamma}$,

ii) $\bar{\epsilon}(t) = \epsilon_0 r(t) t^{-\rho}$, for $\rho = \frac{1+c}{2+c}$, $c > \frac{1}{\gamma}$, and $\forall t$: $0 < c_1 \leq r(t) < c_2$.

Theorem 3.2 Under the assumptions of theorem (3.1) and (6′);

$$\lim_{\epsilon_0 \to 0} \sup_{t\geq 0} E \mid x_{\bar{\epsilon}(t)} - y_{\bar{\epsilon}(t)} \mid^2 = 0.$$

Proof: Apply the change of variables: $t = \frac{1}{\epsilon_0}\tau^{2+c}$, $dt = \frac{1}{\epsilon_0}(2+c)\tau^{1+c}d\tau$, to equation (3.1):

$$\dot{x}_\epsilon(\tau,\omega) = \tau^{-\rho(2+c)} r(\tau^{2+c}) H_\epsilon\left(x_\epsilon,\omega,\tau^{2+c}/\epsilon_0\right)(2+c)\tau^{1+c}$$

$$= r(\tau^{2+c}) H_\epsilon(x_\epsilon,\omega,\tau/\epsilon(\tau)),$$

for $\epsilon(\tau) = \epsilon_0 \tau^{-(1+c)}$. Now observe that ϵ satisfies condition (6) in theorem (3.1), which gives the desired result. $\Diamond$

As can be seen from the proof, ρ has to satisfy the conditions $\frac{1}{2} < \rho \leq 1$, and $\bar{\epsilon}(t)$ has to be greater than t^{-1} so that $r(t) \geq c_0 > 0$, which allows the invocation of the previous theorem. It follows that if $\bar{\epsilon}(t) = t^{-1}$, a convergence is assured for any Type II mixing. Obviously, ρ may be larger than 1 since $\bar{\epsilon}$ may be split into two functions, one bounded and the other satisfying the conditions of the theorem. The same argument holds for $r(t)$, however, it is clear that one would like $\bar{\epsilon}$ to go as slow as possible to zero, since then if the averaged version has a limit, the convergence rate of both equations to that limit is inversely proportional to ρ.

It is possible to extend the theory to the cases where the partial derivatives of H have a polynomial growth it time. Then ϵ has to decrease faster so that the above integrals may still be controlled. We get the following theorem:

Theorem 3.3 Assume that B_1, B_2, B_3, and B_4 are bounded by t^α for some $\alpha \geq 0$ in condition 4 of theorem 3.1, and replace condition 6 with the following:

6. $\exists\, \gamma > 0$, $c > \frac{1}{\gamma}$, such that $\varphi(\delta) \leq \delta^{-\gamma}$, and $\bar{\epsilon}(t) \leq t^{-(1+c+3\alpha)}$, for a monotone decreasing $\bar{\epsilon}$. Then

$$\lim_{\epsilon_0 \to 0} \sup_{t \geq 0} E \mid x_\epsilon(t) - y_\epsilon(t) \mid^2 = 0.$$

Proof: When applying the lemma as before we get the following:

$$I = \sum_k O(\delta_k)(k-1)^{2\alpha}$$

$$II = \sum_k O(\varphi(\delta_k/\epsilon(k-1))k^{2\alpha}$$

$$III = \sum_k O(\delta_k)k^{3\alpha}.$$

Now chose $\delta_k = \sqrt{\epsilon_0}(k-1)^{-(1+\frac{1}{2}(c-\frac{1}{\gamma})+3\alpha)}$, then since $\epsilon(t) \leq t^{-(1+c+3\alpha)}$, we get just as before $\delta_k/\epsilon(k-1) \geq \frac{1}{\sqrt{\epsilon_0}}(k-1)^{\frac{1}{2}(c-\frac{1}{\gamma})}$. The rest of the proof follows exactly as before. $\Diamond$

Extending theorem 3.2 to the case where the partial spatial derivatives are bounded by a polynomial in t is done by absorbing the growth of H into ϵ, which gives the following corollary:

Corollary 3.4 Assume that B_1, B_2, B_3 and B_4 are bounded by t^α for some $\alpha \geq 0$ in condition 4 of theorem 3.1, and replace condition 6 in theorem 3.2 with the following:

6'. $\exists \gamma > 0$, such that

i) $\varphi(\delta) < \delta^{-\gamma}$,

ii) $\bar{\epsilon}(t) = \epsilon_0 r(t) t^{-(\alpha+\rho)}$, for $\rho = \frac{1+c}{2+c}$, $c > \frac{1}{\gamma}$, and $\forall t$: $0 < c_1 \leq r(t) < c_2$. Then

$$\lim_{\epsilon_0 \to 0} \sup_{t \geq 0} E \mid x_{\bar{\epsilon}(t)} - y_{\bar{\epsilon}(t)} \mid^2 = 0.$$

An important observation has to be made here: If the deterministic version represents a converging trajectory, e.g., if the equation represents a gradient descent, then as long as $\bar{\epsilon}(t) \geq t^{-1}$, the deterministic version will still converge to a true local minimum, however if $\bar{\epsilon}(t) < t^{-1}$, then $\int_0^\infty \bar{\epsilon}(\tau) < \infty$, and so the convergence of the deterministic equation is not assured, which implies that the convergence of the stochastic version to a true local minimum is not granted.

4. An application to the synaptic modification equations of a BCM neuron

In this section, we apply the theorem to a random differential equation representing the low governing synaptic weight modification in the BCM theory for learning and memory in neurons, Bienenstock et al. (1982). We start with a short review on the notations and definitions of BCM theory, a more thorough review can be found in Intrator (1990), and the references therein.

Consider a neuron whose input is the vector $x = (x_1, \ldots, x_N)$, has a synaptic-weight vector $m = (m_1, \ldots, m_N)$, both in R^N, and activity (in the linear region) $c = x \cdot m$. The input x is assumed to be a stochastic process of Type II φ mixing, bounded, and piecewise

constant. Let $\Theta_m = E[(x \cdot m)^2]$, $\phi(c, \Theta_m) = c^2 - \frac{4}{3}c\Theta_m$. c represents the linear projection of x onto m, and we seek an optimal projection in some sense.

The BCM synaptic modification equations are given by:

$$\dot{m} = \mu(t)\phi(x \cdot m, \Theta_m)x, \quad m(0) = m_0, \tag{4.1}$$

their averaged version is given by:

$$\dot{\bar{m}} = \mu(t)E\Big[\phi(x \cdot \bar{m}, \Theta_{\bar{m}})x\Big], \quad \bar{m}(0) = m_0. \tag{4.2}$$

$\mu(t)$ is a global modulator which is assumed to take into account all the global factors affecting the cell, e.g., the beginning or end of the critical period, or state of arousal (Bear and Cooper, 1988).

Equation (4.2) is shown to be a dimensionality reduction method based on a cost function that favors directions m for which the distribution of the inputs is different from normal by means of skewness (Intrator, 1990).

Our aim is to show the convergence of the stochastic differential equation. This will be done in two step; First we show that the averaged deterministic equation converges, and then we use theorem 3.2 to show the convergence of the random differential equation to its averaged deterministic equation.

The convergence of the deterministic equation

Without loss of generality, we may assume that the random process x is in the unit ball in R^N, and $\mathrm{Var}(x \cdot m) \geq \lambda \| m \|^2 > 0$, which simply says that x does not lie in a subspace or a manifold of R^N. Since we are interested in dimensionality reduction, we can always reduce a-priori the dimensionality of x so that it will span R^N for some N. When the theory is applied to a finite value random vector, $x_1, \ldots, x_n$, we can restrict m to be in the span of $x_1, \ldots, x_n$.

When we multiply both sides of the above equation by $\bar{m}_\mu$, assuming none of its components is zero, we get:

$$\begin{aligned}\frac{1}{2}\parallel \dot{\bar{m}}_\mu \parallel &= E[(x\cdot\bar{m}_\mu)^3] - \frac{4}{3}E^2[(x\cdot\bar{m}_\mu)^2] \\ &\le \parallel \bar{m}_\mu \parallel^3 - \frac{4}{3}\mathrm{Var}^2(x\cdot\bar{m}_\mu) \\ &\le \parallel \bar{m}_\mu \parallel^3 - \frac{4}{3}\lambda^2 \parallel \bar{m}_\mu \parallel^4 \\ &= \parallel \bar{m}_\mu \parallel^3 \{1 - \frac{4}{3}\lambda^2 \parallel \bar{m}_\mu \parallel\},\end{aligned}$$

which implies that $\parallel \bar{m}_\mu \parallel \le \frac{3}{4\lambda^2}$. ◇

Using this fact we can now show the convergence of $\bar{m}_\mu$. We observe that $\dot{\bar{m}}_\mu = -\nabla R$, where $R(\bar{m}_\mu) = -\frac{\mu}{3}\{E[(x\cdot\bar{m}_\mu)^3] - E^2[(x\cdot\bar{m}_\mu)^2]\}$ is the risk. R is bounded from below since $\parallel \bar{m}_\mu \parallel$ is bounded, therefore $\bar{m}_\mu$ converges to a local minimum of R. ◇

The convergence of the stochastic equation

Claim Under the above conditions $m_\mu(t)$ converges in L^2 to a local minimum of the risk.

Proof: The calculation above implies that $\bar{m}_\mu$ is bounded for (almost) every μ.

In our case B_1, B_2, B_3 and B_4 are independent of t or m_μ, therefore, if we replace $\epsilon(t)$ by $\mu(t)$ and apply theorem 3.2, we get

$$\sup_{t\ge 0} E|m_\mu(t) - \bar{m}_\mu(t)|^2 \underset{\mu_0\to 0}{\longrightarrow} 0.$$

$\bar{m}_\mu$ the solution to the deterministic equation will converge to the same local minimum $\tilde{y}$, $\forall\mu$ if $\mu_0 < C$, for some positive constant C. therefore we can choose $\tilde{T}$ for which $|\bar{m}_\mu(t) - \tilde{y}| < \frac{\delta}{2}$, $\mu_0 < C$, $t \ge \tilde{T}$, then for $t \ge \tilde{T}$ we have:

$$|m_\mu(t) - \tilde{y}| \le |m_\mu(t) - \bar{m}_\mu(t)| + |\bar{m}_\mu(t) - \tilde{y}| \le |m_\mu(t) - \bar{m}_\mu(t)| + \frac{\delta}{2},$$

$$\Rightarrow \quad \sup_{t\ge\tilde{T}} E|m_\mu(t) - \tilde{y}| \le \sup_{t\ge\tilde{T}} E|m_\mu(t) - \bar{m}_\mu(t)| + \frac{\delta}{2} \underset{\mu_0\to 0}{\longrightarrow} \frac{\delta}{2}.$$

δ is arbitrary, which implies that

$$E|m_\mu(t) - \tilde{y}| \underset{\mu \to 0}{\longrightarrow} 0$$

$\Diamond$

5. Summary

It has been shown that under mild conditions, the equations $\dot{x}_\epsilon = \epsilon H(x, \omega, t)$, and $\dot{y}_\epsilon = \epsilon G(y, t)$ where $G(x, t) = E[H(x, \omega, t)]$, have close trajectories in the infinite interval when $\epsilon(t) \leq t^{-\frac{1}{2}}$. The result may be computationally useful, and as has been shown in the example, may assist in the analysis of the random differential equation.

Acknowledgments It is a pleasure for me to thank Stuart Geman for suggesting and guiding me through this problem, and to Yali Amit for many helpful remarks. Research was funded in part by the Office of Naval Research, the Army Research Office, the Air Force office of Scientific Research, and the National Science Foundation.

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